# Using lexicography for learning mathematics

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**Abstract**

Students struggle with the transition from school to university in mathematics. One reason is that at school, mathematics tends to be presented as an ensemble of calculations rather than as a network of concepts. We plan to investigate how lexicography and e-dictionary construction can help students in this transition. In the paper, we introduce the concept of a seminar that uses lexicographic methods in first-year mathematics courses. In the seminar, students will be provided with basic lexicographic knowledge and thus enabled to discuss the newly learned concepts and the relations that hold between them. We also present the lexicographic concept of the resource to be developed in the course: We focus on its article structure and its access structure and describe both in terms of the function theory of lexicography. We suggest innovative access structures which can support the acquisition of mathematical concepts as well as of mathematical terminology. The article structures are based on an ontology structure of the subject matter domain with different kinds of concepts and relations between them.

**Keywords:** terminology; learning; mathematics

## 1. Introduction

In mathematics, students struggle especially with the transition from school to university (Geisler & Rolka, 2021). One of the reasons might be that, at school, mathematics tends to be presented as an ensemble of calculations rather than concepts. Thus, students have to learn that mathematics is basically a building constructed of definitions, theorems, and relations between them. We plan to investigate to which extent a lexicographic approach to e-dictionary construction can help in this transition.

In this paper, we present the concept of a seminar accompanying a regular lecture for first-year students in mathematics. In the seminar, the students collaboratively create a lexical resource on the concepts and terminology that they learn in the lecture. In the following, we discuss lexicographic methods as well as the design of the lexical resource to be created in the seminar.

Our contribution shows the concept of the planned dictionary as well as the structure of the seminar which is intended to accompany an introductory lecture in mathematics. In Sections 2 and 3, we present related work and the subject matter area. In Section 4, we describe the prerequisites based on the function theory. In Section 5, we present the lexicographic concept, and in Section 6, the concept of our planned seminar. We conclude in Section 7.

## 2. Related Work

About twenty years ago, Cubillo (2002) already used lexicography with chemistry students. To support them in understanding and learning concepts of chemistry as well as the pertaining terminology, they were invited to create their own (printed) dictionaries of the field. However, this exercise was not backed up by any lexicographic introduction or training. Since then, electronic dictionaries took root and almost replaced printed dictionaries in several fields (cf. Fuertes-Olivera, 2016).

Kruse & Heid (2020) present a concept of how to structure the mathematical terminology of graph theory for a lexicographic purpose. They establish the following conceptual categories: types of graphs (e.g. *Petersen graph*), parts of graphs (e.g. *edge*, *node*), properties (e.g. *bipartite*), activities (e.g. *(to) map*), theorems (e.g. *four color theorem*), mappings (e.g. *isomorphism*), algorithms (e.g. *Dijkstra's algorithm*). Between concepts of these classes, one or more of the following relations may hold: equivalence, synonymy, hypernymy/hyponymy, holonymy/meronymy, pertonymy, antonymy, mediality, analogy, alternative, attributivity, mapping, eponymy. A similar inventory of concept types and relations may be used in our project.

The lexicographic function theory was developed over several years and is presented by Tarp (2008) and Fuertes-Olivera & Tarp (2014) in its current form. The theory provides a framework to describe the usage situations of a dictionary based on the users' needs. The users can be characterized by their lexicographic knowledge, their terminological knowledge, their expertise level in a special field, and their language level. The users can be in communicative, cognitive, operative, or interpretative situations. The combination of user-profiles and situations leads to different needs which can be fulfilled by a dictionary and which motivate the dictionary design. Below, we analyze the lexicographic needs of first-year mathematics students and thus motivate our dictionary design (cf. Section 4).

Tall & Vinner (1981) and Vinner (1991) introduce the theory of concept image and concept definition in the didactics of mathematics. The concept image denotes non-verbal associations a learner has with a certain term. These associations are always influenced by personal experience and thus continuously re-shaped. It is difficult to exactly determine the concept image of a learner for a particular concept. It can only be expressed by the concept definition, i.e. by how a learner verbalizes a certain concept. Concept image and concept definition always interact. Following this theory, learning is the process of the development and the evolution of concept images and concept definitions. An electronic dictionary might support students in this process as it contains concept definitions that contribute to shaping the concept images.

## 3. Subject Matter Area

We focus on a lecture that is a general introduction to mathematics, intended for teacher students. In this course, students learn the concepts of algebraic structures like groups, rings, and fields as well as vector spaces and matrices. Aspects of these concepts are also used in engineering, economics, and natural sciences.

In the following, we introduce some mathematical concepts which are the basis for the examples used in Section 5. As the introduction of all the axioms necessary to properly introduce the concepts from a mathematical perspective goes beyond the scope of this paper, we rather give a general description of the concepts.

A *relation* is – mathematically speaking – a set of ordered pairs. We use some examples to illustrate what that means. Common examples are the less-than-relation $<$, the divisibility-relation $|$ or the equality-relation $=$. We show some properties such relations can have. The first property we look at is *reflexivity* which means that the relation exists between two same elements, which only applies for divisibility and equality as $a|a$ and $a = a$ but not for less-than as $a < a$ is not true. Symmetry means that, if the relation holds for $a$ and $b$, it also holds for $b$ and $a$. This is only true for the equality as from $a = b$ it follows that $b = a$, but it is not true for less-than as $a < b$ does not imply $b < a$, and not for divisibility because $a|b$ only implies $b|a$ if $a = b$ but not in general. Another common property is transitivity which holds for all three examples: From $a < b$ and $b < c$, one can conclude that $a < c$ and similarly $a|b$ and $b|c$ implies that $a|c$; finally $a = b$ and $b = c$ implies $a = c$. If a relation is reflexive, symmetric, and transitive it is called an *equivalence relation*, which is only the case for equality in our examples.

Further possible properties of relations are among others *left-total*, *right-total*, *left-unique*, and *right-unique*. We do not discuss them here in detail as they require a broader mathematical basis but we introduce some terminology which is derived from these concepts. A *function* as taught in high school is a left-total and right-unique relation. If the function is also right-total it is called *surjective* and if it is left-unique it is called *injective*. If the function is surjective and injective it is called *bijective*. These terms are used in the examples in Figures 1 and 2 in Section 5.

## 4. Intended Usage Situations

In introductory university courses in mathematics, students have to learn concepts, the relations between them, and typical phrases of the specialized language of mathematics. At school, however, mathematics tends to be presented as an ensemble of calculations rather than concepts. Thus, students have to learn that mathematics is basically a building constructed of definitions, theorems, and relations between them. The course at hand consists of a lecture and related tutorial lessons. At the end of the course, students have to pass a written examination. Each year about 150 students have to attend the lecture in the first year of their teacher program. In the following, we describe the students' needs by relying on the function theory by Tarp (2008) and Fuertes-Olivera & Tarp (2014).

The intended users speak German at a first language level as they are studying in a German Bachelor's program. We also assume that they have an advanced level of English due to their school education. We regard them as laypeople in both, their mathematical concept knowledge and their mathematical language knowledge, as they are in their first year of study. Even if they have reached a certain degree in school mathematics which gives them useful background knowledge, we can safely assume this categorization, as academic mathematics highly differs from school mathematics in most cases.

Furthermore, we assume them to be acquainted with using online resources as general sources of information but are only beginning to rely on lexicographic tools for mathematics, as such resources are not commonly used in mathematical school education. While Wikipedia as a kind of lexicographic tool is often used by students in mathematics

(Henderson et al., 2017; Anastasakis & Lerman, 2022), we assume that they only begin using it in the course of their studies as Wikipedia presents mathematics in a way it is taught at universities but not in schools. As we work with first-year students, we have the possibility of changing or even shaping their habits in our seminar: Investigations show that Wikipedia articles do not always provide the highest quality information (Jayakody & Zazkis, 2015; Selwyn & Gorard, 2016; Dunn et al., 2019) and that they may not always be easy to understand for non-experts (Kruse & Heid, 2022).

The learners are in our case mainly in a systematic cognitive situation following the terminology by Tarp (2008) and Fuertes-Olivera & Tarp (2014). There might be smaller sporadic cognitive situations as well as short communicative situations but we neglect the latter two for our conceptualization as the main goal of the course is to provide mathematical knowledge, i.e. shaping the concept image as well as learning the corresponding concept definitions from a formal perspective. Cognitive situations with the need to consult an electronic dictionary might thus occur in the following ways: attending a lecture, watching a learning video, discussing with fellow students, working on tasks, or reading a script or a textbook.

The concept image not only consists of discrete concepts but certain relations occur between them on a conceptual level and are expressed in the concept definitions as semantic relations on a linguistic level. In a formal domain like mathematics, these two levels of relations are almost completely identical. Nevertheless, linguistic relations between terms also appear, e.g. synonymy: Several expressions denote the same abstract concept and should be presented by the same concept image. For example, students have to learn that the symbols $\{\}$, $\emptyset$ and the term *empty set* all refer to the same concept, namely a set without any elements in it.

From these user prerequisites as well as their usage situations the following user needs evolve which should be fulfilled by lexicographic assistance: The most common need is to look up the definition of a given term. In this context, not only the formal definition but also further information on the usage of the term is useful, i.e. in the form of concrete examples. In some cases, users might need algorithms for carrying out certain calculations, e.g. the Euclidean algorithm to find the greatest common divisor of two natural numbers.

A similar need affects not only one but two terms as users might be interested in their relation; for example, if they denote the same concept (e.g. *node* and *vertex*) or if they exclude each other (e.g. *positive integers* and *negative integers*). Conversely, it might be the case that a user has the right concept in mind but does not know the term which is used for it. Another example need is that users contextualize definitions in the concepts they have already learned, e.g. a *tree* is defined as a graph that does not contain any cycles. The learning of the new concept *tree* requires knowledge of the concepts *graph* and *cycle*. From a user perspective, it might be interesting to find out for two given terms if their combination yields a new term. In all these cases, the dictionary should be able to provide assistance.

## 5. Lexicographic concept

Based on the users, their situations, and their needs, we present a dictionary concept, in particular regarding the article structure and the access structure. Further, based on the idea that the dictionary content is developed by the students, it is a semi-collaborative dictionary at the first stage which might be used as a resource with only indirect user involvement later on by other students (cf. Abel & Meyer, 2016).

### 5.1 Article structure

The content given in an article (i.e. in a dictionary entry) depends on the type of the particular lemma. Building on the work by Kruse & Heid (2020) we use four concept categories: OBJECT, PROPERTY, THEOREM, METHOD. OBJECT comprises all kinds of mathematical entities with mappings, parts, and types as sub-categories. Examples are *set* or *group*. In the category PROPERTY, we comprise all properties these entities could have, e.g. *complete* or *bijectivity*. THEOREM are all kinds of mathematical statements, like propositions, lemmas, or theorems themselves. For our conceptualization, we do not differentiate if the theorem has been proven yet. The theorems make statements about the elements of the categories PROPERTY and OBJECT. The last category is called METHOD and comprises algorithms as well as mathematical strategies for proving.

Between the elements of the categories, different semantic relations exist. Some of them have been already pointed out in the description of the categories. The relations can exist between members of the same and of different categories. We work with the following relations:

- OBJECT$_1$ is hypernym of OBJECT$_2$
- OBJECT can have PROPERTY
- OBJECT has always PROPERTY
- THEOREM is about OBJECT
- THEOREM is about PROPERTY
- THEOREM$_1$ implies THEOREM$_2$
- METHOD is based on THEOREM
- METHOD can find OBJECT (with PROPERTY)
- METHOD$_1$ and METHOD$_2$ have same goal
- PROPERTY$_1$ implies PROPERTY$_2$
- PROPERTY$_1$ excludes PROPERTY$_2$
- PROPERTY$_1$ and PROPERTY$_2$ can co-exist

The list above is also visualized in Table 1. It should be read by starting with one of the items in the leftmost column; the item above the relation field is the second object of the relation; e.g. *THEOREM is about OBJECT*. The table only covers relations on the conceptual level. Further relations on the linguistic level can appear, like synonymy. Additionally, between two entities from the category OBJECT more relations than indicated here are possible like holonymy/meronymy or antonymy. The selection criteria that define which of them should be included in the dictionary will be developed in the seminar (cf. Section 6)

In an article, the names of concepts that are in a certain relation to the lemma are given in addition to the definition. For example for an OBJECT, the dictionary article gives the following information: hypernyms, hyponyms, facultative properties, mandatory properties, theorems about it, and methods how to calculate it. To avoid overloading the mathematics students with lexicographical terminology, we suggest using the general

| | Method | Object | Property | Theorem |
|---|---|---|---|---|
| Method | has same goal as | can find | can find | is based on |
| Object | can be found by | is hypernym/hyponym of/... | can be / is always | is mentioned in |
| Property | can be found by | is always attached to / can be attached to | implies / excludes / can co-exist with | is mentioned in |
| Theorem | is basis for | is about | is about | implies |

Table 1: Possible relations between concepts to be indicated in the microstructure

language paraphrases of the relations given above and using them directly as structural indicators.

It needs to be decided and evaluated how the article should be presented: For example, if it should be shown in one of the rather classical electronic views like panel view, tab view, explorer view or print view (Koplenig & Müller-Spitzer, 2014) or if more innovative forms should be used which give a better visualization of the network-like structure of mathematical conceptualizations (e.g. by means of knowledge graphs), as proposed in EcoLexicon (cf. e.g. León-Araúz et al., 2019). In Figure 1, we show such a presentation, focused on a single lemma, namely the term *equivalence relation*. It might also be possible to let users switch between different view formats as it might depend on the user and their particular need in a given situation which view fits best.

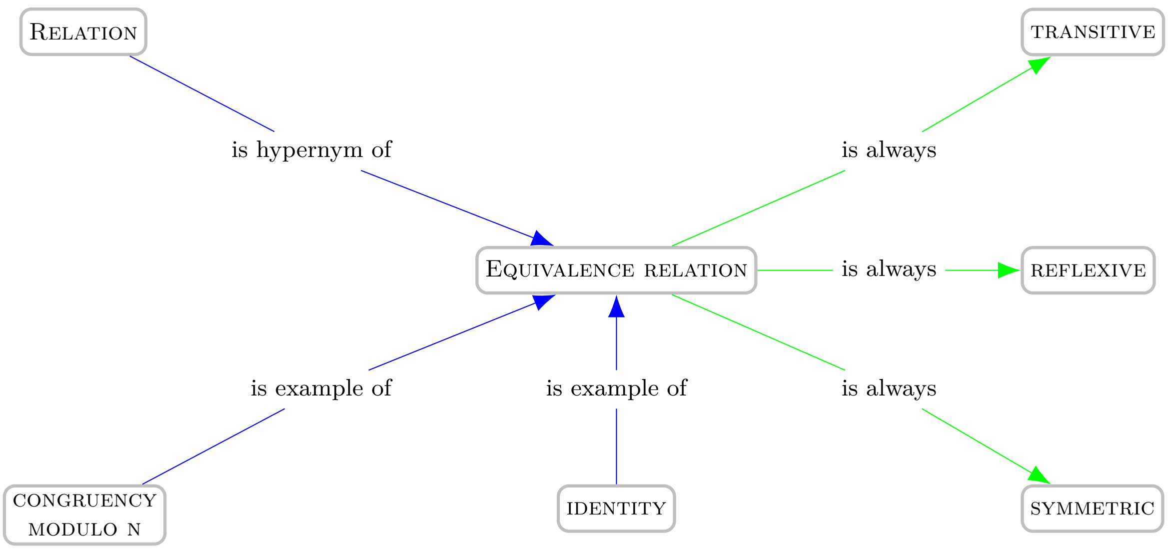


Figure 1: Example article for *Equivalence relation*

In addition to these conceptual categories, there is a category of domain-specific phraseology (e.g. *if and only if*, *q.e.d.*, *corollary*). But as this cannot be really integrated into the concept net it should be provided as part of the outer features.

### 5.2 Access structure and access paths

To satisfy the user needs we described in Section 4, likely more than one access structure will be needed, and students may use several types of access paths in combination. The example of the graph-based article structure can be the starting point of a graph-based access structure. It should allow users to zoom in and out of the graph. In addition to the graph-based access structure, an input-based search should be implemented as well a navigation.

The input-based search can be used to find definitions and examples for a given term. In other cases, the navigation or the graph-based search are probably useful aids. For example, if someone has a concept in mind but does not know (or does not remember) the appropriate term, they can navigate through the graph until they arrive at the right term. In some cases also a full-text search might help as well as an access structure using general language. To that end, the names of the concepts can be associated internally with quasi-synonyms from general language which allow users to find them. A search for *is equal to* or *is the same as* could then point the user to lemmas such as *isomorphic*, *identical*, or *equivalent*. Either by the graph-based structure or by the navigation it should be also possible to name two concepts and get the relation between them as a result. An example of such an excerpt from the concept net is shown in Figure 2.

## 6. Seminar concept

The dictionary as it is conceptualized here is not isolated but integrated into the lecture, as it is a task for the students to write articles of this semi-collaborative dictionary. Additionally, they can have their own private dictionary each, comparable to an individual flashcard set. Thus, the writing of the articles is a fixed part of the seminar accompanying the lecture. This individual student work is accompanied by sessions of the seminar in which the students can discuss their results.

We plan to give the students basic lexicographic training and access to a dictionary writing system that is optimized for the construction of specialized dictionaries, in particular for mathematics. Therein, they can note the concepts they have learned and indicate the semantic relations between these concepts. In the seminar, the students also learn basic lexicographic knowledge to be able to appropriately use the provided tool.

When building their personal e-dictionaries during the course, we introduce the students to a routine for including new terms:

1. Collect the new terminology and phraseology from your lecture notes and from the literature you worked with last week.
2. Choose a category for each term. If there are theorems that only have a number but no name, choose an appropriate name for them.
3. Find relations between the new concepts from the established relations.
4. Connect the new terms to the ones already learned.

If there are terms the students have difficulties allocating a category to, this will be discussed in the seminar. This empirical validation helps to improve the category system.

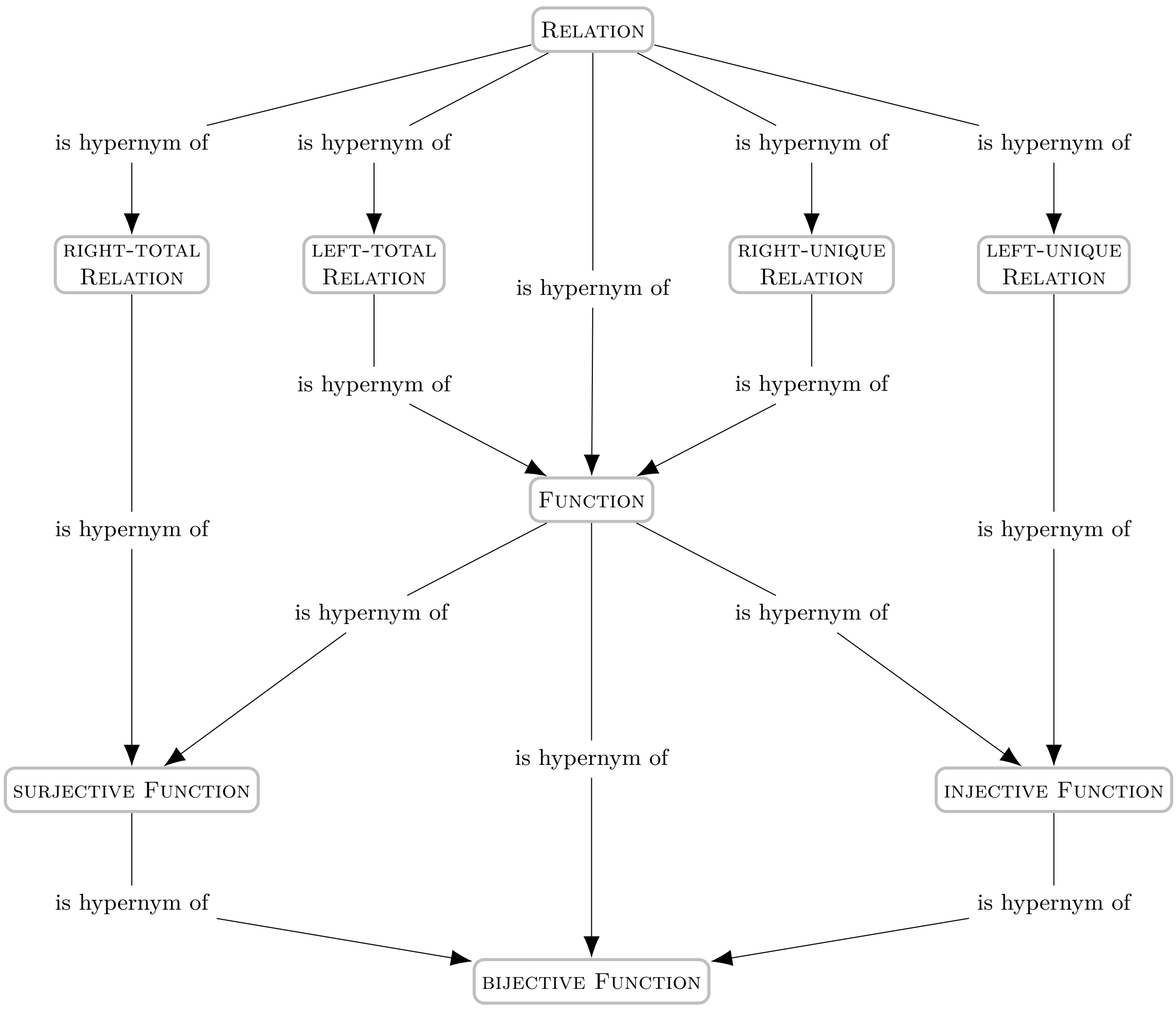


Figure 2: Extract from the network of concepts

New categories may be added to the conceptualization. The same applies to difficulties in assigning the relations between the terms.

Concurrently, the lexicographic structuring of the data helps the students to gain a deeper understanding of mathematics which in turn supports the acquisition of the content as it addresses the constructivist dimension of learning (Girnat & Hascher, 2021).

The dictionary writing system to be used has to fulfill certain requirements for the project. It needs to be easy to use as the students should be able to focus on learning the mathematical concepts rather than being distracted by the software. This also implies the inclusion of mathematical formulae by clicking or drag-and-drop as not all of the students – especially in the first year – have enough knowledge in scientific word processing, e.g. with LaTeX. Additionally, it should be possible to search through the entries but also to navigate through them by use of the categories and relations, in order to use them in other articles. Further possible extensions are the export of flashcards and tagging for individual learning progress. We aim at an open source framework to be independent of economic interests and to allow students to continue using the system in the further course of their studies.

## 7. Conclusion and future work

In this paper, we presented a concept for a lexicographic resource that can be used in the process of learning mathematics. As a next step, we will implement a prototype of such a resource and use it in a lecture and a seminar with students to evaluate it. The implementation of the dictionary tool will likely be done by using existing frameworks that can be combined with the learning platform used in the courses. Choosing and establishing an appropriate system is the next step in the project.

Concerning the evaluation, we plan to compare the students in our proposed seminar with a group of students who attended a regular seminar with the same content. Both groups will be tested on their mathematical knowledge as well as on their mathematical beliefs (Pehkonen & Törner, 1996).

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